A (virtual) double category theorist's perspective on POLYNOMIALS

Nathanael Arkor   Bryce Clarke

Topos Colloquium 03.03.25

Polynomials

A **polynomial** in a category $\mathcal{E}$ is a diagram:

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$$

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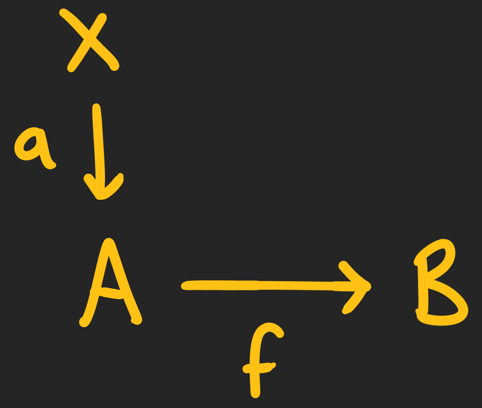
Polynomials provide simple presentations of a well-behaved class of functors — the polynomial functors — in such a way that many useful constructions on polynomial functors are reflected in natural constructions on polynomials.

Distributivity_pullbacks

A pullback around (a, f)



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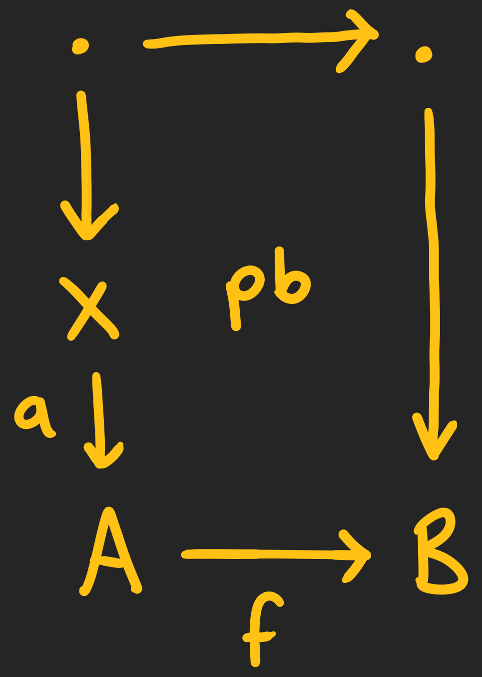

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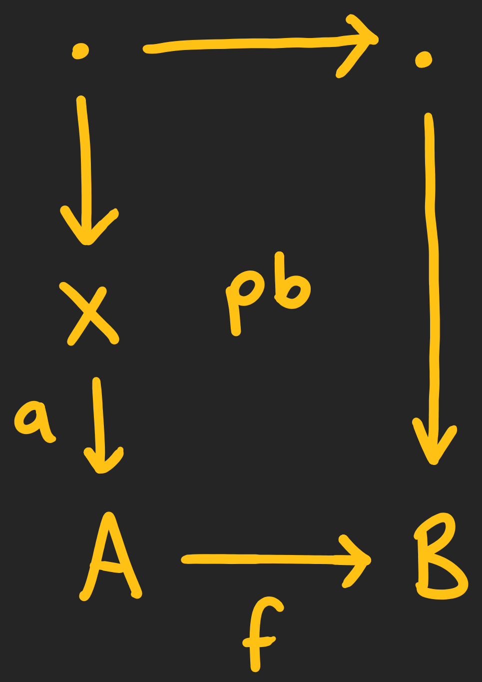
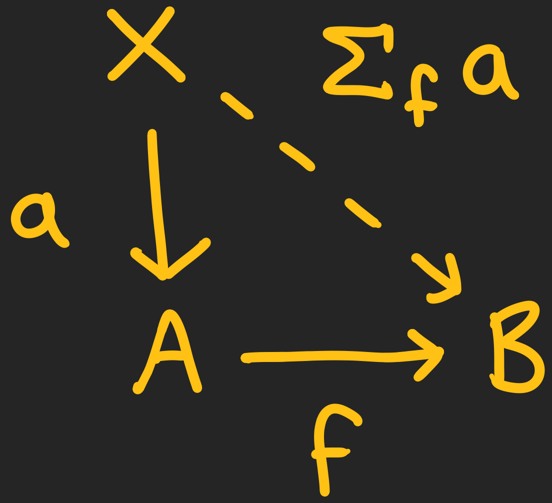


A distributivity pullback is a terminal such diagram.

Three operations on slices

Let $A \xrightarrow{f} B$ be a morphism in a category $\mathcal{E}$.

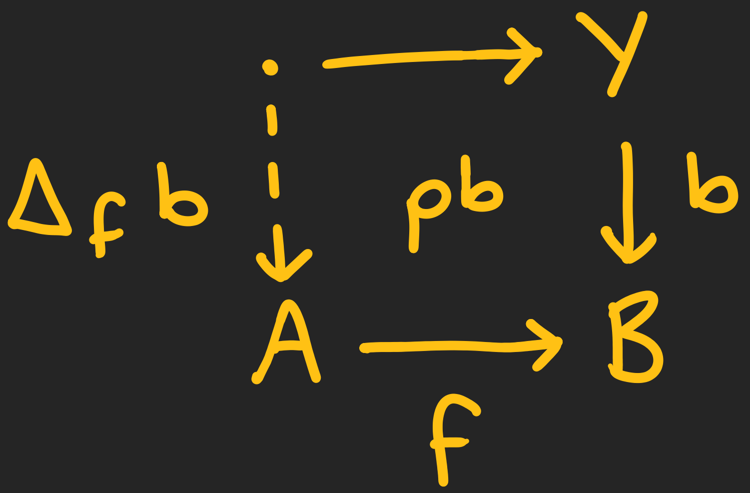
① Given a morphism into A, we define:



Three operations on slices

Let $A \xrightarrow{f} B$ be a morphism in a category $\mathcal{E}$.

(2) Given a morphism into B, we define:



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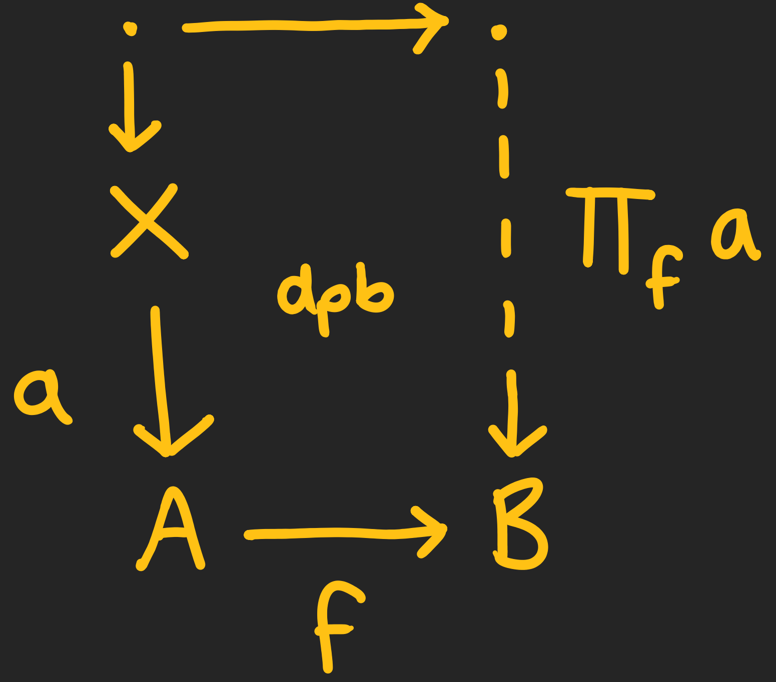

Locally cartesian closed categories

A category is LCC if it admits pullbacks and distributivity pullbacks. Every morphism $A \xrightarrow{f} B$ in a LCCC $\mathcal{E}$ induces three functors:

$$\mathcal{E}/B \xrightarrow{\Delta_f} \mathcal{E}/A \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_f} \mathcal{E}/B$$

These functors satisfy $\Sigma_f \dashv \Delta_f \dashv \Pi_f$.

Polynomial functors

Every polynomial in a LCC category $\mathcal{E}$

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$$

induces a functor:

$$\mathcal{E}/I \xrightarrow{\Delta_s} \mathcal{E}/B \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_t} \mathcal{E}/J$$

A functor is polynomial if it is isomorphic to one induced by a polynomial.

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1. If polynomial functors are so prevalant, is there any sense in which polynomials are canonical?
2. What is the formal status of the assignment of polynomial functors to polynomials?
3. Can polynomials be used to present any other structures?

Linear polynomials

A **span** in a category is a diagram:

$$I \xleftarrow{\ s\ } A \xrightarrow{\ t\ } J$$

Linear polynomials

A **span** in a category is a diagram:

$$I \xleftarrow{s} A \xrightarrow{t} J$$

Spans may be viewed as simple polynomials:

$$I \xleftarrow{s} A = A \xrightarrow{t} J$$

To understand polynomials, it may help to first understand spans.

Where do spans come from?

In what sense is the concept of a span canonical?

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In what sense is the concept of a span **canonical**?

In category theory, to show something is canonical means exhibiting a **universal property** involving it.

To do so, we shall need some preliminary concepts.

Double categories

A **double category** $\mathbb{D}$ comprises:

- a category $\mathbb{D}_0$ of **objects** and **tight morphisms** $A \longrightarrow B$

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- a category $\mathbb{D}_0$ of **objects** and **tight morphisms** $A \longrightarrow B$
- for each pair of objects A, B, a collection of **loose morphisms** $A \nrightarrow B$
- for each frame $\begin{array}{ccc} A & \overset{p}{\nrightarrow} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \underset{p'}{\nrightarrow} & B' \end{array}$, a collection of **cells**

- for each object A, an identity

$$A \overset{}{\Relbar\!\!\!\!\!\mid\!\!\!\Relbar} A$$

- for loose morphisms $A \overset{p}{\nrightarrow} B$ and $B \overset{q}{\nrightarrow} C$, a composite

$$A \overset{p \odot q}{\nrightarrow} C$$

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$$A \xrightarrow{\;p \odot q\;} C$$

- identity cells:

$$\begin{array}{ccc} A & \overset{p}{\nrightarrow} & B \\ \Vert & = & \Vert \\ A & \underset{p}{\nrightarrow} & B \end{array} \qquad\qquad \begin{array}{ccc} A & \mathrel{=\!\!|\!\!=} & A \\ a\downarrow & = & \downarrow a \\ A' & \mathrel{=\!\!|\!\!=} & A' \end{array}$$

- composites of cells:

$$
\begin{array}{ccccc}
A & \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & B & \overset{q}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & C \\
{\scriptstyle a}\downarrow & \varphi & \downarrow & \psi & \downarrow{\scriptstyle c} \\
A' & \underset{p'}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & B' & \underset{q'}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & C'
\end{array}
\quad\longmapsto\quad
\begin{array}{ccc}
A & \overset{p \odot q}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & C \\
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\end{array}
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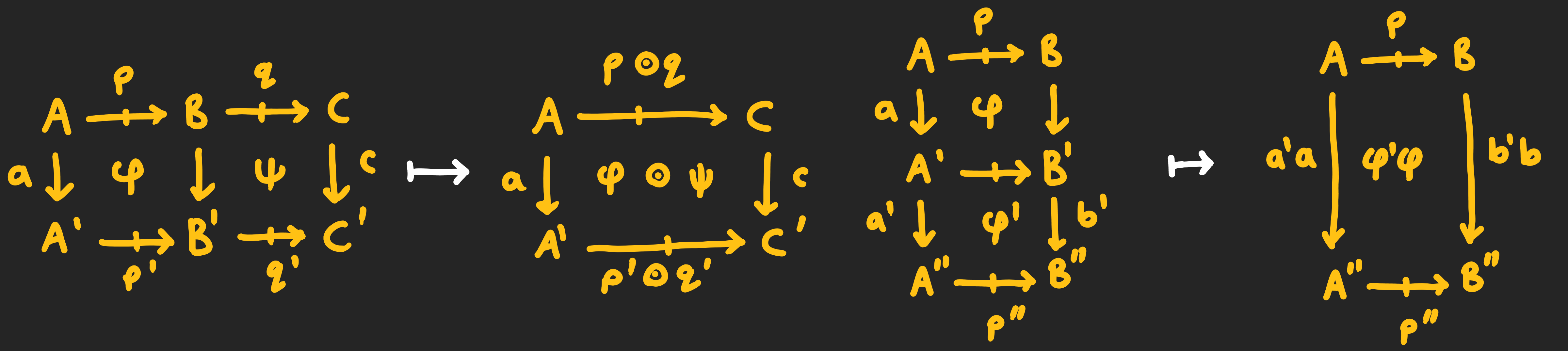
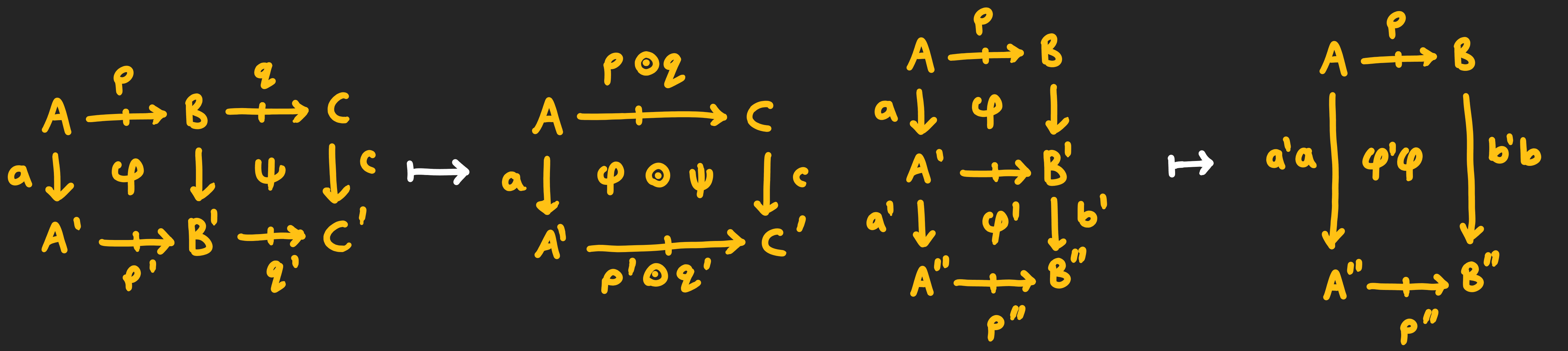
A' & \longrightarrow & B' \\
a'\downarrow & \varphi' & \downarrow b' \\
A'' & \underset{p''}{\longrightarrow} & B''
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
A & \overset{p}{\longrightarrow} & B \\
a'a\downarrow & \varphi'\varphi & \downarrow b'b \\
A'' & \underset{p''}{\longrightarrow} & B''
\end{array}
$$

- composites of cells:



- Structural isomorphisms expressing associativity and unitality of loose composition.

- composites of cells:



- Structural isomorphisms expressing associativity and unitality of loose composition.

These data are subject to various unsurprising axioms.

The double category of spans

For every category $\mathcal{E}$ with pullbacks, there is a double category $\mathbb{S}\mathrm{pan}(\mathcal{E})$ whose underlying category is $\mathcal{E}$, in which a loose morphism $I \nrightarrow J$ is a span

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and in which a cell (left) is a morphism (right):

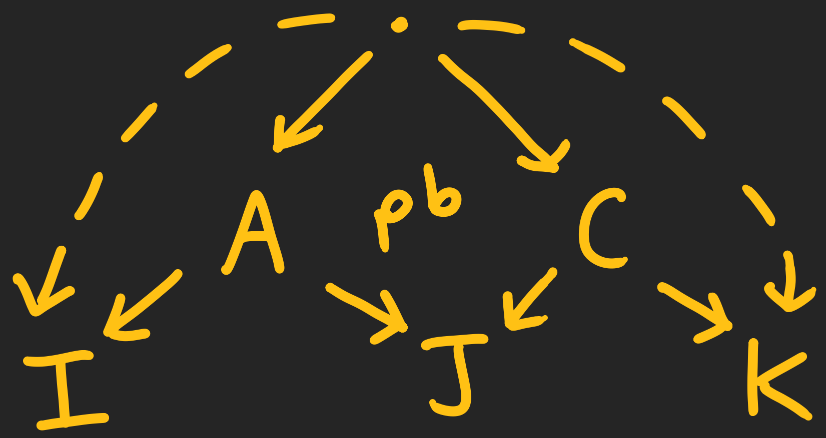
$$\begin{array}{ccc} I & \xrightarrow{(s,t)} & J \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ I' & \xrightarrow[(s',t')]{} & J' \end{array} \qquad \begin{array}{ccccc} I & \xleftarrow{s} & A & \xrightarrow{t} & J \\ {\scriptstyle i}\downarrow & = & \downarrow & = & \downarrow{\scriptstyle j} \\ I' & \xleftarrow[s']{} & A' & \xrightarrow[t']{} & J' \end{array}$$

The double category of spans

The identity span on an object A is:

$$A = A = A$$

The composite of $I \leftarrow A \rightarrow J$ and $J \leftarrow C \rightarrow K$ is:



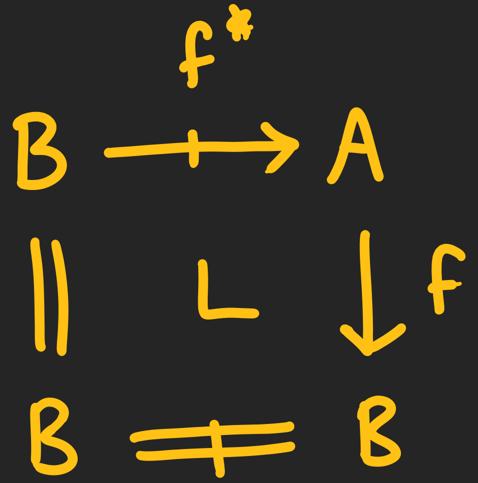
Companions

A **companion** for a tight morphism $A \xrightarrow{f} B$ is a loose morphism $A \xrightarrow{f_*} B$ equipped with cells

satisfying the zig-zag laws ( = 1 ,  = —).

Conjoints

A **conjoint** for a tight morphism $A \xrightarrow{f} B$ is a loose morphism $B \xrightarrow{f^*} A$ equipped with cells



satisfying the zig-zag laws ().

Adjoints

An adjunction in a double category is a pair of loose morphisms, together with cells

satisfying the zig-zag laws (∩∪ = 1, ∪∩ = 1).

Companions & conjoints

Lemma Any two of the following implies the third.

- f admits a companion and a conjoint
- f admits a companion and $f_* \dashv$
- f admits a conjoint and $\dashv f^*$

In this case, $f_* \dashv f^*$.

Adjoint pairs in $\mathbb{S}\mathrm{pan}(\mathcal{E})$

Every tight morphism $A \xrightarrow{f} B$ in $\mathbb{S}\mathrm{pan}(\mathcal{E})$ admits a companion and conjoint.

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Every tight morphism $A \xrightarrow{f} B$ in $\mathbb{S}\mathrm{pan}(\mathcal{E})$ admits a companion and conjoint.

$$\left(A \overset{=}{\longleftarrow} A \overset{f}{\longrightarrow} B\right) \dashv \left(B \overset{f}{\longleftarrow} A \overset{=}{\longrightarrow} A\right)$$

Furthermore, every span $I \xleftarrow{s} A \xrightarrow{t} J$ is a composite:

$$\left(I \overset{s}{\longleftarrow} A \overset{=}{\longrightarrow} A\right) \odot \left(A \overset{=}{\longleftarrow} A \overset{t}{\longrightarrow} J\right)$$

Gregarious double categories

A double category is gregarious if every tight morphism admits a companion and a conjoint.

Gregarious double categories are also called equipments.

The universal property of $\mathbb{S}\mathrm{pan}(\mathcal{E})$

Theorem (Dawson–Paré–Pronk)

For each category $\mathcal{E}$ with pullbacks, $\mathbb{S}\mathrm{pan}(\mathcal{E})$ is the free gregarious double category on $\mathcal{E}$.

$$\mathrm{GregDbl}_c(\mathbb{S}\mathrm{pan}(\mathcal{E}), \mathbb{D}) \simeq \mathrm{Cat}(\mathcal{E}, \mathbb{D}_0)$$

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Conceptually, spans arise by freely adjoining companions (right leg) and conjoints (left leg).

From spans to polynomials

Double categories provide a convenient context in which to understand how spans arise canonically.

Can we apply similar reasoning to understand how polynomials arise canonically?

The double category of polynomials

For every LCC category $\mathcal{E}$, there is a double category $\mathbb{P}\mathrm{oly}(\mathcal{E})$ whose underlying category is $\mathcal{E}$, in which a loose morphism $I \nrightarrow J$ is a polynomial

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$$\begin{array}{ccc} I & \xrightarrow{(s,f,t)} & J \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ I' & \xrightarrow[(s',f',t')]{} & J' \end{array} \qquad\qquad \begin{array}{ccccccc} I & \leftarrow & B & \rightarrow & A & \rightarrow & J \\ \downarrow & & & & \vdots & = & \downarrow \\ I' & \leftarrow & B' & \rightarrow & A' & \rightarrow & J' \end{array}$$

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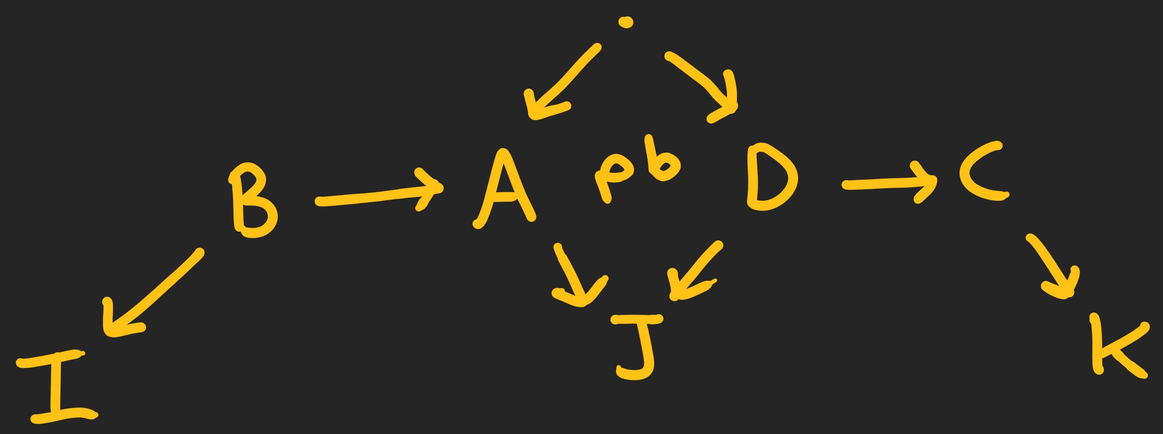

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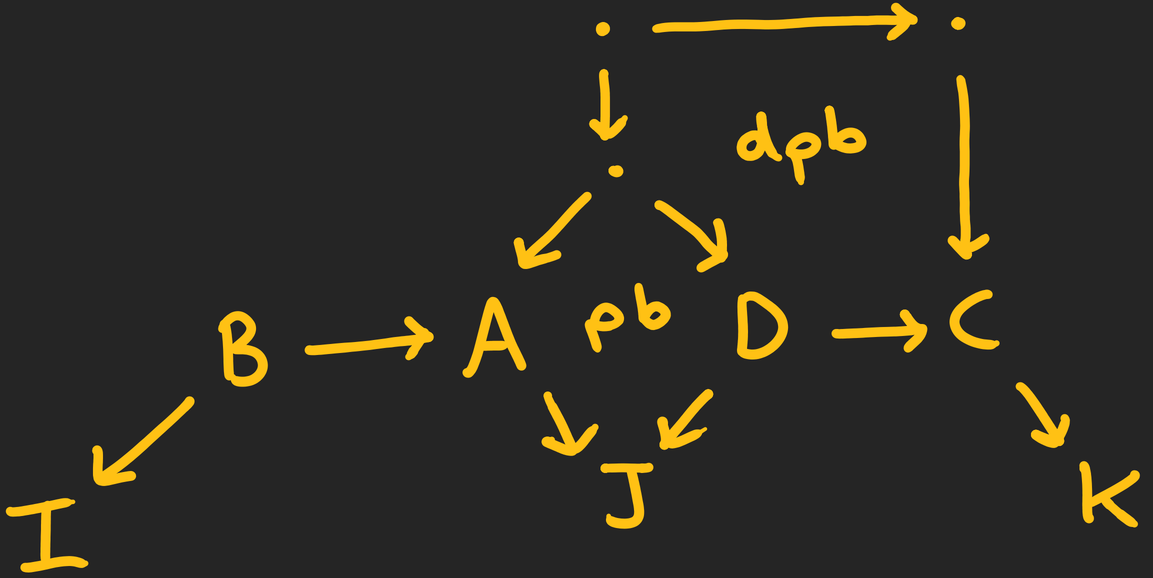

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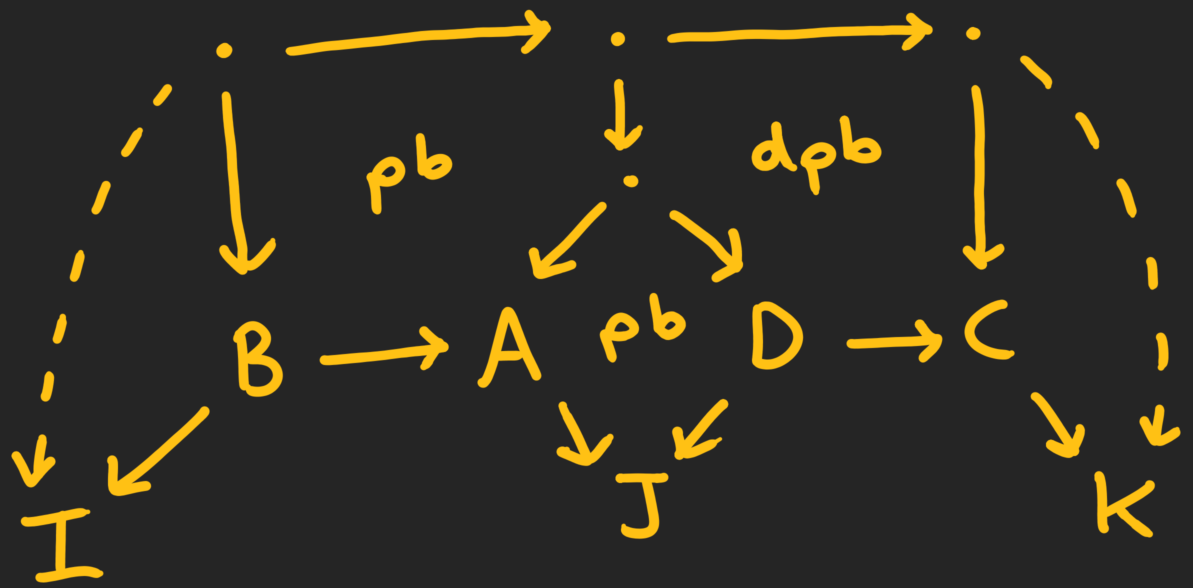

Polynomials are gregarious

Theorem (Gambino–Kock)

$\mathbb{P}\mathrm{oly}(\mathcal{E})$ is gregarious: every tight morphism $A \xrightarrow{f} B$ admits a companion and conjoint.

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Corollary There is a canonical normal colax functor $\mathbb{S}\mathrm{pan}(\mathcal{E}) \to \mathbb{P}\mathrm{oly}(\mathcal{E})$.

Correlates

Recall that, in a double category with conjoints, a tight morphism $A \xrightarrow{f} B$ admits a companion if its conjoint $B \overset{f^*}{\nrightarrow} A$ admits a <u>left</u> adjoint.

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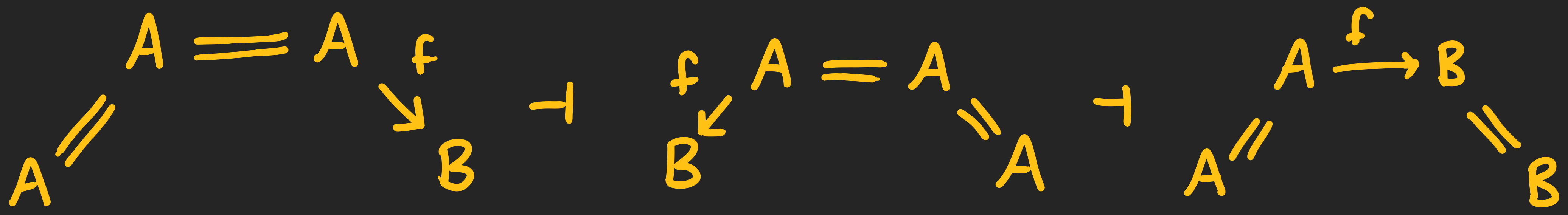
Recall that, in a double category with conjoints, a tight morphism $A \xrightarrow{f} B$ admits a companion if its conjoint $B \xrightarrow{f^*} A$ admits a left adjoint.

A correlate for f is a right adjoint to f^*. We denote it $A \xrightarrow{f_!} B$.

Adjoint triples in $\mathbb{P}\mathrm{oly}(\mathcal{E})$

Every tight morphism $A \xrightarrow{f} B$ in $\mathbb{P}\mathrm{oly}(\mathcal{E})$ admits a companion, conjoint, and correlate.



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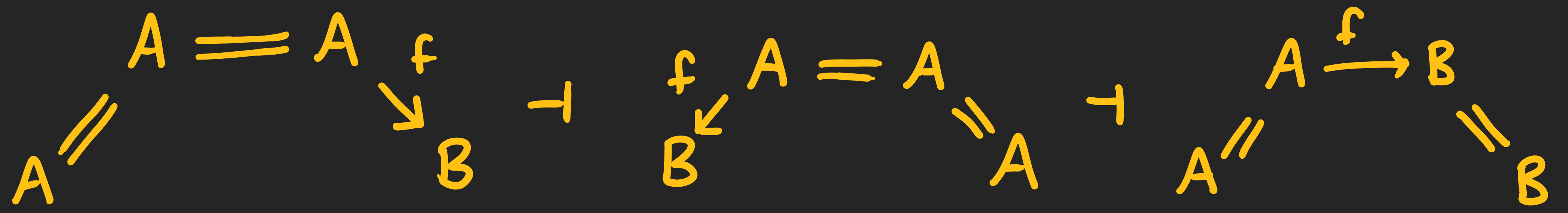
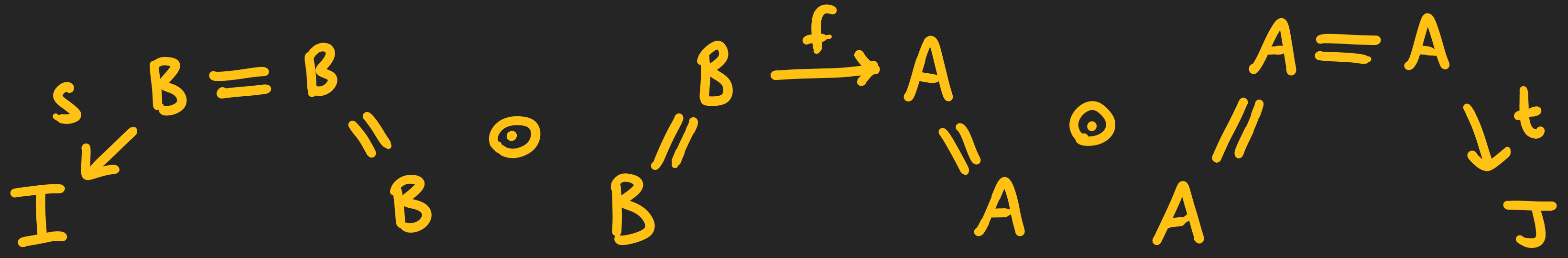


Furthermore, every polynomial $I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$ is a composite:



A conjectural universal property

Just as spans are canonically built by composing companions and conjoints, polynomials are canonically built by composing companions, conjoints, and correlates.

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Just as spans are canonically built by composing companions and conjoints, polynomials are canonically built by composing companions, conjoints, and correlates.

We might therefore conjecture that $\mathbb{P}\mathrm{oly}(\mathcal{E})$ is the free double category with companions, conjoints, and correlates. Unfortunately, the complex definition makes this tricky to verify.

Bridges in double categories

A **bridge** in a double category is a diagram:

$$I \xrightarrow{p} A \xrightarrow{t} J$$

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$$I \overset{p}{\nrightarrow} A \xrightarrow{t} J$$

For instance, a bridge in $\mathbb{S}\mathrm{pan}(\mathcal{E})$ is exactly a polynomial in $\mathcal{E}$.

$$I \leftarrow B \rightarrow A \xrightarrow{t} J$$

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We wish to construct a double category in which loose morphisms are bridges. How should we compose?

Bridgeable double categories

A double category $\mathbb{D}$ is **bridgeable** when:

- $\mathrm{dom} : \mathbb{D}_1 \longrightarrow \mathbb{D}_0$ is a fibration
- $\mathrm{Id} : \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ is a morphism of fibrations

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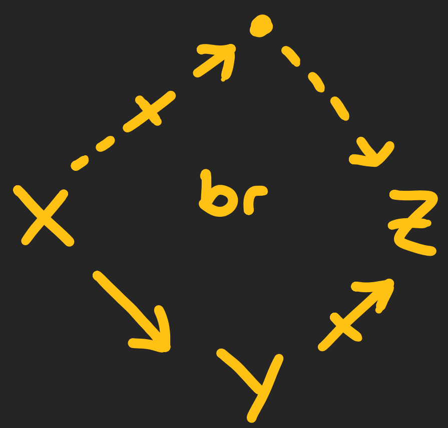
Intuitively, this means that given any diagram there exists a bridge and a cell **br** that is universal in an appropriate sense.



The double category of bridges

For each bridgeable double category $\mathbb{D}$, there is a double category $\mathbb{B}\mathrm{r}(\mathbb{D})$ with the same underlying category for which:

- a loose morphism is a bridge in $\mathbb{D}$
- a cell comprises

$$\begin{array}{ccccc} I & \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & A & \overset{t}{\longrightarrow} & J \\ {\scriptstyle i}\downarrow & \varphi & \vdots\!\downarrow & = & \downarrow{\scriptstyle j} \\ I' & \underset{p'}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & A' & \underset{t'}{\longrightarrow} & J' \end{array}$$

The double category of bridges

The identity bridge on an object A is:

$$A \mathrel{\rlap{\,|}{=}} A = A$$

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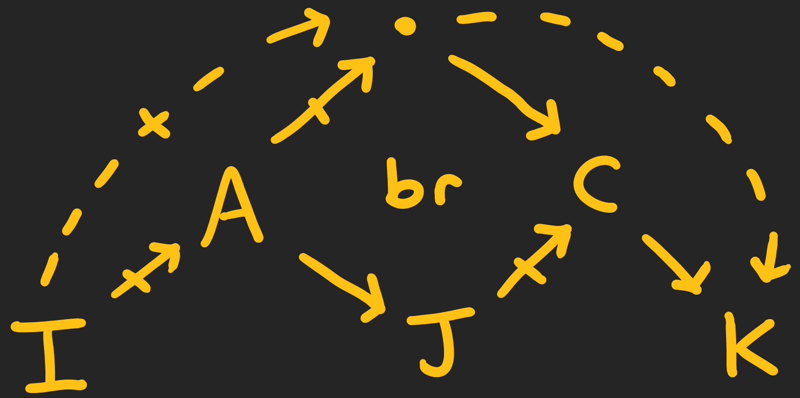


Polynomials are bridges where?

We know that a bridge in $\mathbb{S}\mathrm{pan}(\mathcal{E})$ is a polynomial. We might therefore expect $\mathbb{P}\mathrm{oly}(\mathcal{E})$ to arise as $\mathbb{B}\mathrm{r}(\mathbb{S}\mathrm{pan}(\mathcal{E}))$.

However, a cell here would be

$$\begin{array}{ccccccc} I & \leftarrow & B & \rightarrow & A & \rightarrow & J \\ \downarrow & = & \downarrow & = & \downarrow & = & \downarrow \\ I' & \leftarrow & B' & \rightarrow & A' & \rightarrow & J' \end{array}$$

which is not the definition of polynomial morphism.

Coretrocells

In general, it doesn't make sense to reverse the cells in a double category, because if we flip a cell, the tight morphisms face in the wrong direction:

$$
\begin{array}{ccc}
A & \overset{p}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & B \\
{\scriptstyle a}\big\downarrow & \varphi & \big\downarrow{\scriptstyle b} \\
A' & \underset{p'}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & B'
\end{array}
\qquad \longmapsto \qquad
\begin{array}{ccc}
A' & \overset{p'}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & B' \\
{\scriptstyle a}\big\uparrow & ? & \big\uparrow{\scriptstyle b} \\
A & \underset{p}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & B'
\end{array}
$$

Coretrocells

However, in a double category with conjoints, there is a bijection between cells with the following frames:

$$
\begin{array}{ccc}
A & \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & B \\
a \big\downarrow & \varphi & \big\downarrow b \\
A' & \underset{p'}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & B'
\end{array}
\qquad\qquad
\begin{array}{ccccc}
A' & \overset{a^*}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & A & \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & B \\
\| & & \overline{\varphi} & & \| \\
A' & \underset{p'}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & B' & \underset{b^*}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & B
\end{array}
$$

We <u>can</u> flip cells of the shape on the right.

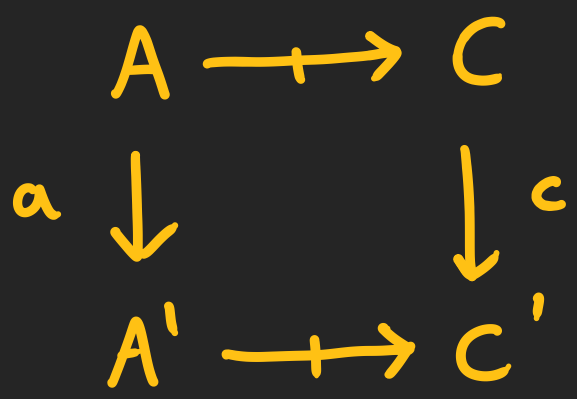
Coretrocells

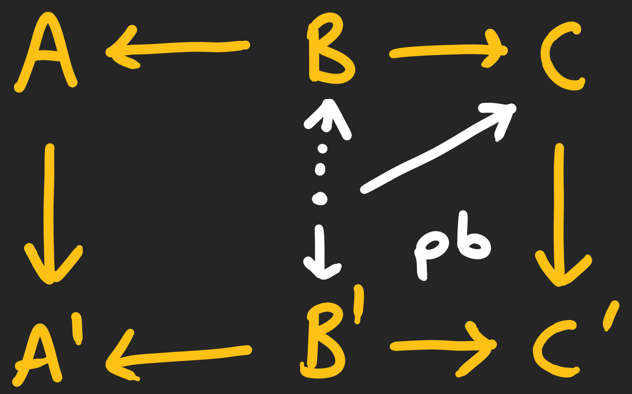
For each double category $\mathbb{D}$ with conjoints, there is a double category $\mathbb{D}^{crt}$ with the same underlying category and loose morphisms, but for which a cell

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle a}\downarrow & \varphi & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[p']{} & B' \end{array} \quad \text{in } \mathbb{D}^{crt} \quad \text{is} \quad \begin{array}{ccccc} A' & \xrightarrow{p'} & B' & \xrightarrow{b^*} & B \\ \| & & \varphi & & \| \\ A' & \xrightarrow[a^*]{} & A & \xrightarrow[p]{} & B \end{array} \quad \text{in } \mathbb{D}.$$

Span coretromorphisms

For a category $\mathcal{E}$ with pullbacks, $\mathbb{S}\mathrm{pan}(\mathcal{E})^{crt}$ is the double category whose underlying category is $\mathcal{E}$, whose loose morphisms are spans in $\mathcal{E}$, and in which a cell (left) is given by (right):





Polynomials as bridges

Theorem (AC)

For a category $\mathcal{E}$ with pullbacks, $\mathbb{B}\mathrm{r}(\mathbb{S}\mathrm{pan}(\mathcal{E})^{\mathrm{crt}})$ is isomorphic to the double category of polynomials in $\mathcal{E}$.

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Notably, the complicated definition of polynomial composition is captured by the significantly simpler composition of bridges.

Companionable double categories

A double category is companionable if every tight morphism admits a companion.

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Example For every bridgeable double category $\mathbb{D}$, the double category $\mathbb{B}r(\mathbb{D})$ is companionable: the companion of $A \xrightarrow{f} B$ is $A \mathrel{+\!\!\!=} A \xrightarrow{f} B$.

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Example For every bridgeable double category $\mathbb{D}$, the double category $\mathbb{B}\mathrm{r}(\mathbb{D})$ is companionable: the companion of $A \xrightarrow{f} B$ is $A \mathrel{\not=} A \xrightarrow{f} B$.

Furthermore, every bridge $I \overset{p}{\nrightarrow} A \xrightarrow{t} J$ is a composite $I \overset{p}{\nrightarrow} A = A \;\circ\; A \mathrel{\not=} A \xrightarrow{t} J$.

The universal property of $\mathbb{B}r(\mathbb{D})$

Theorem (AC)

For each bridgeable double category $\mathbb{D}$, $\mathbb{B}r(\mathbb{D})$ is the free companionable double category on $\mathbb{D}$.

$$\mathrm{CompDbl}_c(\mathbb{B}r(\mathbb{D}), \mathbb{E}) \simeq \mathrm{Dbl}_{nc}(\mathbb{D}, \mathbb{E})$$

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Conceptually, bridges arise by freely adjoining companions to existing loose morphisms.

Properties of $\mathbb{D}^{crt}$

Lemma (Paré) $\mathbb{D}^{crt}$ admits conjoints.

Lemma (AC) $\mathbb{D}^{crt}$ admits companions (resp. correlates) iff $\mathbb{D}$ admits correlates (resp. companions).

Properties of $\mathbb{D}^{crt}$

Lemma (Paré) $\mathbb{D}^{crt}$ admits conjoints.

Lemma (AC) $\mathbb{D}^{crt}$ admits companions (resp. correlates) iff $\mathbb{D}$ admits correlates (resp. companions).

Corollary $\mathbb{S}\mathrm{pan}(\mathcal{E})^{crt}$ is the free double category with conjoints and correlates on $\mathcal{E}$.

Putting the pieces together

Lemma

The inclusion $\mathbb{D} \longrightarrow \mathbb{B}r(\mathbb{D})$ sending $I \overset{p}{\nrightarrow} A$ to $I \overset{p}{\nrightarrow} A = A$ is pseudo, and hence preserves conjoints and correlates.

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Lemma

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Corollary

$\mathbb{B}r(\mathbb{S}pan(\mathcal{E})^{crt}) \simeq \mathbb{P}oly(\mathcal{E})$ admits companions, conjoints, and correlates.

The universal property of $\mathbb{P}\mathrm{oly}(\mathcal{E})$

A double category is polynomic if every tight morphism admits a companion, conjoint, and correlate.

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A double category is polynomic if every tight morphism admits a companion, conjoint, and correlate.

Theorem (AC)

For each LCC category $\mathcal{E}$, $\mathbb{P}\mathrm{oly}(\mathcal{E})$ is the free polynomic double category on $\mathcal{E}$.

$$\mathrm{PolyDbl}_c(\mathbb{P}\mathrm{oly}(\mathcal{E}), \mathbb{D}) \simeq \mathrm{Cat}(\mathcal{E}, \mathbb{D}_0)$$

Breaking the construction down

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1. Viewing $\mathcal{E}$ as a double category with trivial loose morphisms and cells, $\mathbb{B}\mathrm{r}(\mathcal{E}) \simeq \mathbb{S}\mathrm{q}(\mathcal{E})$, the double category of squares.

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3. Consequently, $\mathbb{P}\mathrm{oly}(\mathcal{E}) \simeq \mathbb{B}\mathrm{r}(\mathbb{B}\mathrm{r}(\mathbb{B}\mathrm{r}(\mathcal{E})^{\mathrm{opl}})^{\mathrm{crt}})$.

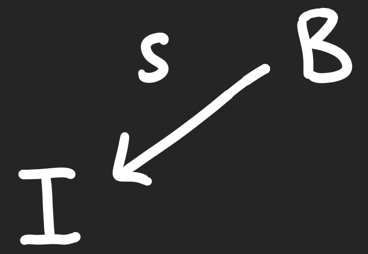
Breaking the construction down

start with the category of interest

$$\mathbb{P}\mathrm{oly}(\Sigma) \simeq \mathbb{B}\mathrm{r}\left(\mathbb{B}\mathrm{r}\left(\mathbb{B}\mathrm{r}(\overbrace{\Sigma})^{opl}\right)^{crt}\right)$$

Breaking the construction down

$$\mathbb{P}\mathrm{oly}(\mathcal{E}) \simeq \mathbb{B}\mathrm{r}\Big(\mathbb{B}\mathrm{r}\big(\overbrace{\mathbb{B}\mathrm{r}(\mathcal{E})^{\mathrm{opl}}}^{\text{first, adjoin conjoints}}\big)^{\mathrm{crt}}\Big)$$



Breaking the construction down

next, adjoin correlates

$$\mathbb{P}\mathrm{oly}(\mathcal{E}) \simeq \mathbb{B}\mathrm{r}\Big(\overbrace{\mathbb{B}\mathrm{r}\big(\mathbb{B}\mathrm{r}(\mathcal{E})^{\mathrm{opl}}\big)^{\mathrm{crt}}}^{\text{next, adjoin correlates}}\Big)$$

$$I \xleftarrow{\;s\;} B \xrightarrow{\;t\;} A$$

Breaking the construction down

$$\mathbb{P}\mathrm{oly}(\mathcal{E}) \simeq \overbrace{\mathbb{B}\mathrm{r}\left(\mathbb{B}\mathrm{r}\left(\mathbb{B}\mathrm{r}(\mathcal{E})^{\mathrm{opl}}\right)^{\mathrm{crt}}\right)}^{\text{finally, adjoin companions}}$$

$$I \xleftarrow{\;s\;} B \xrightarrow{\;t\;} A \xrightarrow{\;u\;} J$$

Breaking the construction down

$$\mathbb{P}\mathrm{oly}(\mathcal{E}) \simeq \mathbb{B}\mathrm{r}\left(\mathbb{B}\mathrm{r}\left(\mathbb{B}\mathrm{r}(\mathcal{E})^{\mathrm{opl}}\right)^{\mathrm{crt}}\right)$$

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Where do polynomials come from?

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2. What is the formal status of the assignment of polynomial functors to polynomials?

The canonical interpretation

For every polynomic double category $\mathbb{D}$, the universal property induces a normal colax functor

$$\mathbb{P}\mathrm{oly}(\mathbb{D}_0) \longrightarrow \mathbb{D}$$

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For every polynomic double category $\mathbb{D}$, the universal property induces a normal colax functor

$$\mathbb{P}\mathrm{oly}(\mathbb{D}_0) \longrightarrow \mathbb{D}$$

sending each polynomial

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$$

in $\mathbb{D}_0$ to its associated polynomial morphism:

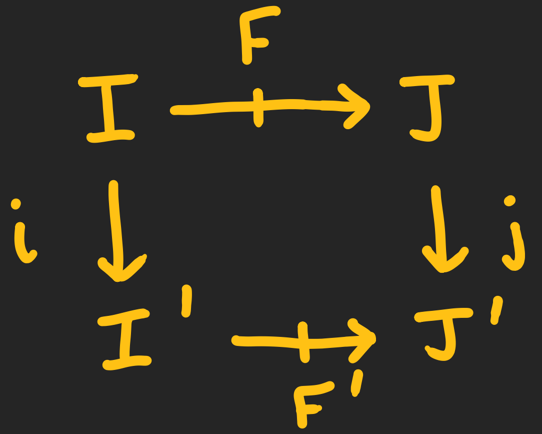
$$I \xrightarrow{s^*} B \xrightarrow{f_!} A \xrightarrow{t_*} J$$

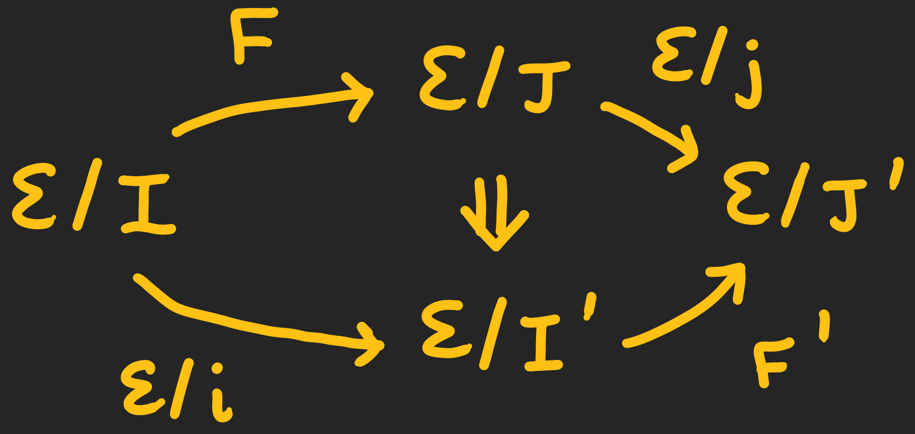
The double category of slices

For every LCC category $\mathcal{E}$, there is a double category $\mathbb{S}\mathrm{lice}(\mathcal{E})$ whose underlying category is $\mathcal{E}$, in which a loose morphism $I \nrightarrow J$ is a functor

$$\mathcal{E}/I \longrightarrow \mathcal{E}/J$$

and in which a cell (left) is a natural transformation (right):





Polynomial functors via universality

$\mathbb{S}\mathrm{lice}(\mathcal{E})$ is polynomic: for a tight morphism $A \xrightarrow{f} B$, the companion, conjoint, and correlate are given by:

$$\Sigma_f \dashv \Delta_f \dashv \Pi_f : \mathcal{E}/A \longrightarrow \mathcal{E}/B$$

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The induced functor $\mathbb{P}\mathrm{oly}(\mathcal{E}) \longrightarrow \mathbb{S}\mathrm{lice}(\mathcal{E})$ is exactly the canonical interpretation of polynomials as polynomial functors.

Where do polynomials come from?

1. If polynomial functors are so prevalant, is there any sense in which polynomials are canonical?
2. What is the formal status of the assignment of polynomial functors to polynomials?
3. Can polynomials be used to present any other structures?

Generalised polynomial functors

Given a polynomial in Cat,

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$$

there is an induced functor

$$\hat{I} \xrightarrow{\hat{s}} \hat{B} \xrightarrow{\mathrm{Ran}_{f^{op}}} \hat{A} \xrightarrow{\mathrm{Lan}_{f^{op}}} \hat{J}$$

between presheaf categories. Such a functor is called a generalised polynomial functor by Fiore.

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between presheaf categories. Such a functor is called a **generalised polynomial functor** by Fiore.

But **Cat** is not LCC, nor need f be exponentiable.

A (virtual) double category theorist's perspective on POLYNOMIALS

Nathanael Arkor   Bryce Clarke

Topos Colloquium 03.03.25

A pullback perplexity

The universal property of $\mathbb{S}\mathrm{pan}(\mathcal{E})$ may appear unusual in that it does not require preservation of pullbacks.

$$\mathrm{GregDbl}_c(\mathbb{S}\mathrm{pan}(\mathcal{E}), \mathbb{D}) \simeq \mathrm{Cat}(\mathcal{E}, \mathbb{D}_0)$$

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The universal property of $\mathbb{S}\mathrm{pan}(\mathcal{E})$ may appear unusual in that it does not require preservation of pullbacks.

$$\mathrm{GregDbl}_c(\mathbb{S}\mathrm{pan}(\mathcal{E}), \mathbb{D}) \simeq \mathrm{Cat}(\mathcal{E}, \mathbb{D}_0)$$

This prompts one to wonder whether pullbacks are necessary at all.

Spans without pullbacks

Pullbacks are required to compose spans. However, we can drop the assumption that we can compose loose morphisms and instead directly describe the data of cells into n-ary composites.

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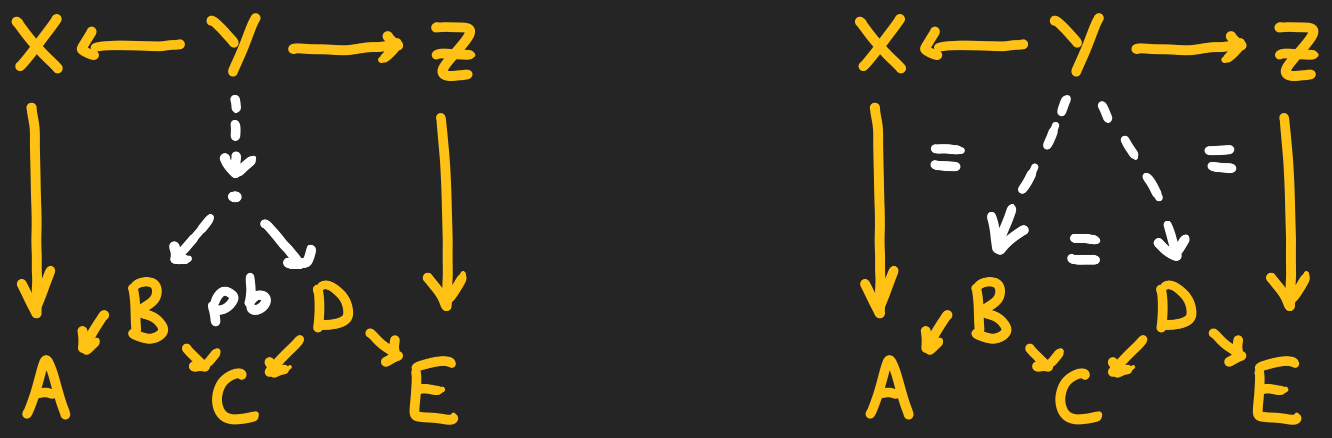
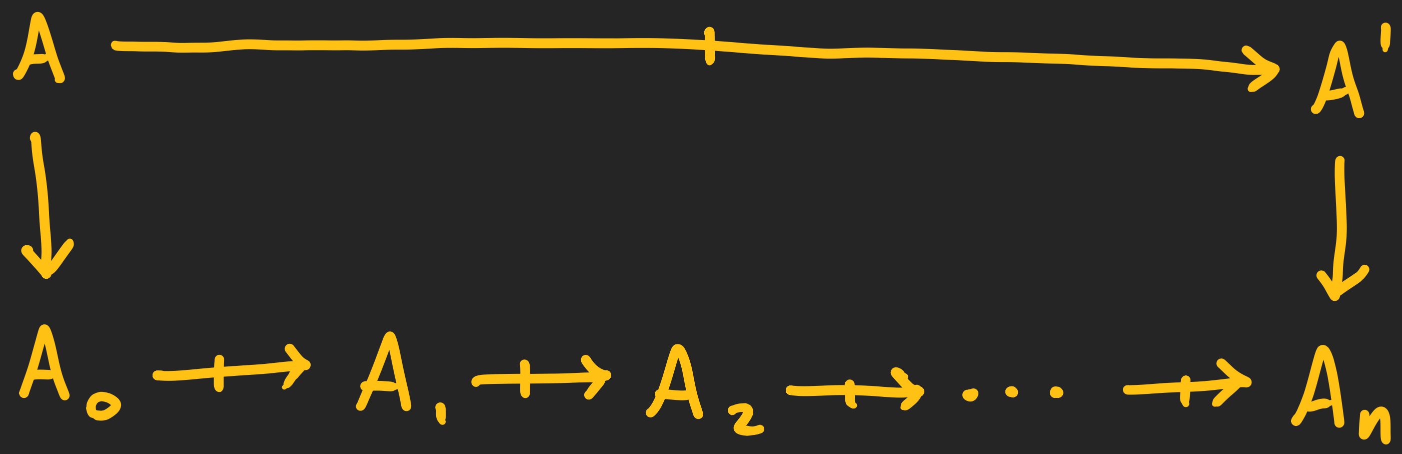
For instance, a span morphism (left) comprises (right):



Covirtual double categories

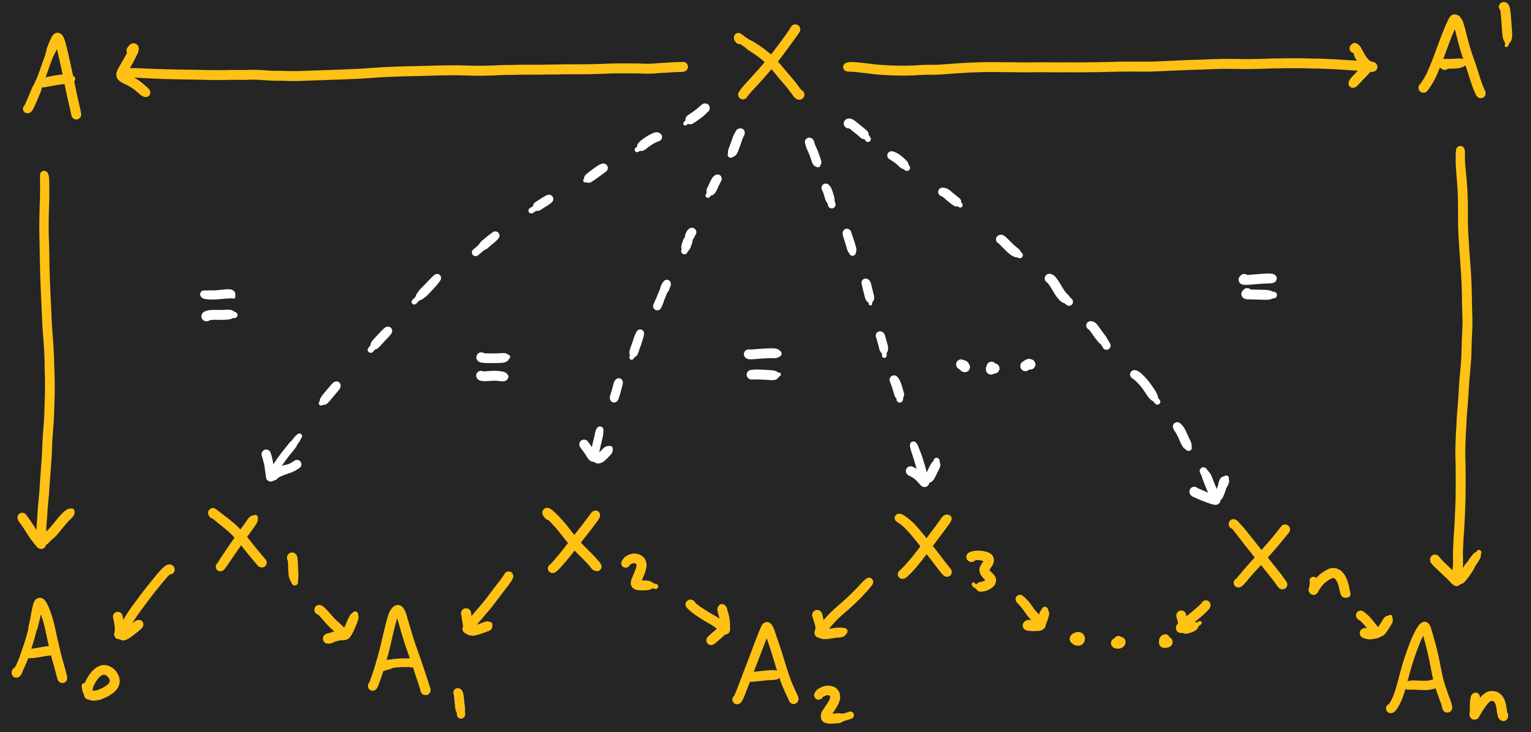
A **covirtual double category** is a structure like a double category, but which does not have identity or composite loose morphisms. Instead, it has **n-ary cells**:



together with suitable compositions of cells.

A covirtual double category of spans

For any category $\mathcal{E}$, there is a covirtual double category $\mathbf{Span}(\mathcal{E})$ in which an n-ary cell comprises:



Gregarious covirtual double categories

We can define what it means for a covirtual double category to admit identity loose morphisms, certain composites, companions, and conjoints.

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A covirtual double category is gregarious if it admits:

- identities
- companions
- conjoints
- composites of the form $s^* \circ t_*$

The universal property of $\mathbb{S}\mathrm{pan}(\mathcal{E})$ II

Theorem (Dawson-Paré-Pronk)

For each category $\mathcal{E}$, $\mathbb{S}\mathrm{pan}(\mathcal{E})$ is the free gregarious covirtual double category on $\mathcal{E}$.

$$\mathrm{GregCovdbl}(\mathbb{S}\mathrm{pan}(\mathcal{E}), \mathbb{D}) \simeq \mathrm{Cat}(\mathcal{E}, \mathbb{D}_0)$$

This frees us from the assumption that $\mathcal{E}$ has pullbacks.

Exponentiability elimination

Can we employ similar techniques to weaken the assumption on $\mathcal{E}$ necessary to form $\mathbb{P}\mathrm{oly}(\mathcal{E})$?

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Pullbacks in $\mathcal{E}$ are necessary to define unary cells, so we cannot drop this assumption.

Exponentiability elimination

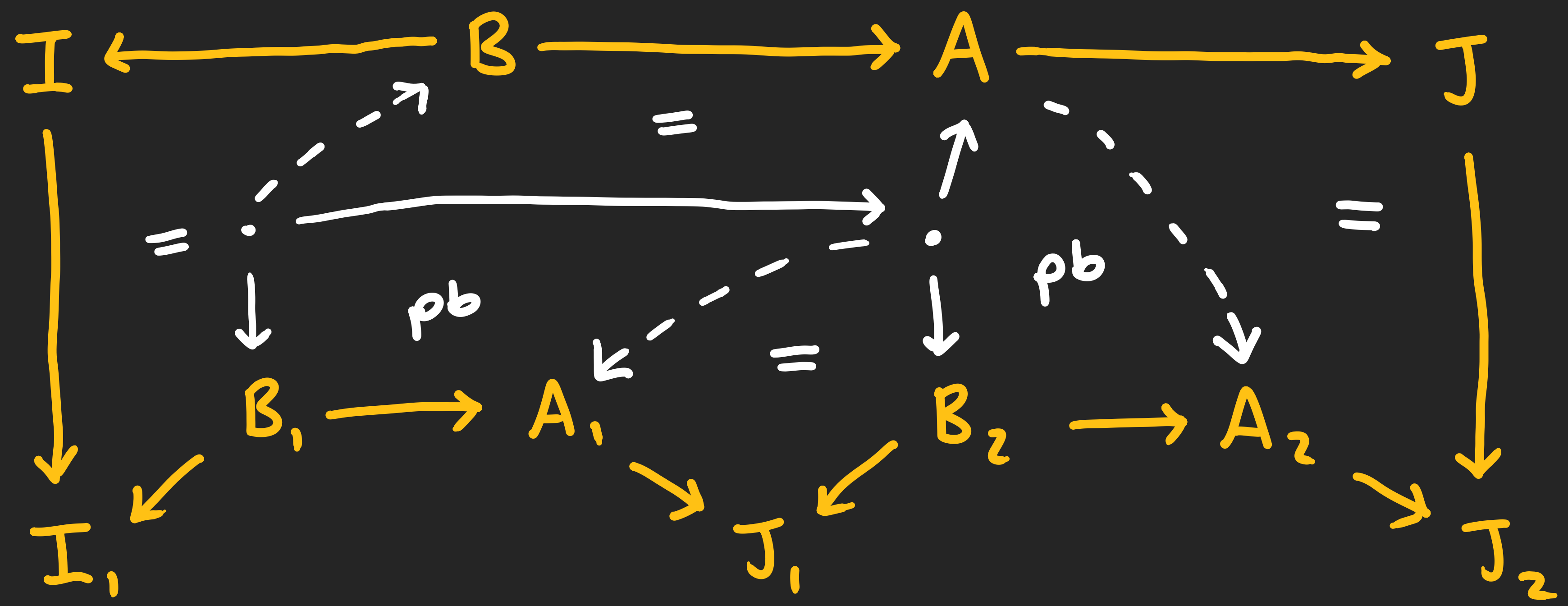
Can we employ similar techniques to weaken the assumption on $\mathcal{E}$ necessary to form $\mathbb{P}\mathrm{oly}(\mathcal{E})$?

Pullbacks in $\mathcal{E}$ are necessary to define unary cells, so we cannot drop this assumption.

However, by unwinding the data of a polynomial morphism into an n-ary composite, we see that it is possible to eliminate distributivity pullbacks.

A covirtual double category of polynomials

For any category $\mathcal{E}$ with pullbacks, there is a covirtual double category $\mathbb{P}\mathrm{oly}(\mathcal{E})$ in which a binary cell is:



Polynomic covirtual double categories

A covirtual double category is polynomic if it admits:

- identities
- companions
- conjoints
- correlates
- composites of the form $s^* \odot f_! \odot t_*$

The universal property of $\mathbb{P}\mathrm{oly}(\mathcal{E})$ II

Theorem (AC)

Let $\mathcal{E}$ be a category with pullbacks. $\mathbb{P}\mathrm{oly}(\mathcal{E})$ is the free polynomic covirtual double category on $\mathcal{E}$.

The universal property of $\mathbb{P}\mathrm{oly}(\mathcal{E})$ II

Theorem (AC)

Let $\mathcal{E}$ be a category with pullbacks. $\mathbb{P}\mathrm{oly}(\mathcal{E})$ is the free polynomic covirtual double category on $\mathcal{E}$.

This frees us from the assumption that $\mathcal{E}$ has distributivity pullbacks / exponentiable morphisms.

The double category of presheaves

There is a double category $\mathbb{P}\mathrm{sh}$ whose underlying category is $\mathbf{Cat}$, in which a loose morphism $I \looparrowright J$ is a functor

$$\hat{I} \longrightarrow \hat{J}$$

and in which a cell (left) is a natural transformation (right):

$$\begin{array}{ccc} I & \overset{F}{\looparrowright} & J \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ I' & \underset{F'}{\looparrowright} & J' \end{array} \qquad\qquad \begin{array}{ccccc} & & \hat{J} & & \\ & \overset{F}{\nearrow} & & \overset{\hat{j}}{\searrow} & \\ \hat{I} & & \Downarrow & & \hat{J'} \\ & \underset{\hat{i}}{\searrow} & & \underset{F'}{\nearrow} & \\ & & \hat{I'} & & \end{array}$$

Generalised polynomial functors via universality

$\mathbb{P}\mathrm{sh}$ is polynomic: for a tight morphism $A \xrightarrow{f} B$, the companion, conjoint, and correlate are given by:

$$\mathrm{Lan}_{f^{op}} \dashv \hat{f} \dashv \mathrm{Ran}_{f^{op}} : \hat{A} \longrightarrow \hat{B}$$

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The induced functor $\mathbb{P}\mathrm{oly}(\mathrm{Cat}) \longrightarrow \mathbb{P}\mathrm{sh}$ is exactly the interpretation of polynomials as generalised polynomial functors.

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Our perspective has been significantly influenced by the following:

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Summary

- The double category of polynomials arises by freely adjoining companions, conjoints, and correlates to a locally cartesian closed category.
- The interpretation of polynomials as polynomial functors arises from this universal property.
- By working with covirtual double categories, we can drop the assumption of exponentiability.